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Type-IIB colliding plane waves

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Abstract: Four-dimensional colliding plane wave (CPW) solutions have played an important role in understanding the classical non-linearities of Einstein's equations. In this note, we investigate CPW solutions in $2n+2$-dimensional Einstein gravity with a $n+1$-form flux. By using an isomorphism with the four-dimensional problem, we construct exact solutions analogous to the Szekeres vacuum solution in four dimensions. The higher-dimensional versions of the Khan-Penrose and Bell-Szekeres CPW solutions are studied perturbatively in the vicinity of the light-cone. We find that under small perturbations, a curvature singularity is generically produced, leading to both space-like and time-like singularities. For $n = 4$, our results pertain to the collision of two ten-dimensional type-IIB Blau-Figueroa o'Farrill-Hull-Papadopoulos plane waves.

Keywords: Penrose limit and pp-wave background, Classical Theories of Gravity.

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1. Introduction

Gravitational colliding plane wave (CPW) solutions have received much attention over the years, as a way to bring insight into the non-linearities of the collision of more realistic gravitational waves. The subject originated in the work of Khan and Penrose [1] and has a vast literature, see [2] for a review and an exhaustive list of references.

CPW may play an important role in primordial cosmology as a possible seed for large scale structure formation or even setting the initial conditions in a pre-big-bang scenario [3]. At a more formal level, colliding gravitational plane waves offer simple models in which to study the fate of the inner null singularity of realistic Kerr black holes [4], and a useful approximation to scattering at Planckian energies [5]. They also receive a cosmological interpretation as Gowdy universes [6] (i.e. with two commuting Killing vectors).

While most of the CPW studies have taken place in the framework of four-dimensional Einstein gravity possibly with an abelian gauge field, it is worthwhile to ask which of these results would continue to apply to higher dimensional gravity theories, and in particular to those which describe the low energy limit of string theory [7]–[11]. In particular, several maximally supersymmetric plane wave solutions have been identified in type-IIB [12] and 11-dimensional supergravity [13]. It is an interesting problem to study their collisions, and whether spacelike singularities are generically produced in such processes. This may be especially tractable due to the high degree of symmetry of these backgrounds. Notice that in order to set-up the collision, it is customary to restrict to waves with bounded support along x^+, thereby breaking part of the supersymmetries.

Finally, while exact CPW solutions of Einstein equations or their supergravity generalizations only pertain to very special initial conditions, it is important to study the stability of plane waves under small perturbations: indeed, plane waves usually come in an infinite dimensional moduli space, corresponding to the x^+-dependent profiles of the various fields restricted by a single Einstein equation. A particle or string traveling along the x^+ direction will generically be transmitted through the wave, but also be partly reflected back and alter the profile of the outgoing wave. This process is usually unaccessible in light-cone quantization approaches, as it involves the emission of $p^+ = 0$ states. It is nevertheless of crucial importance when the background wave presents a null singularity, as it may turn it into a timelike or spacelike singularity. Studying this issue at the level of classical gravity may in particular shed light on the singularities observed in the closed-string parabolic orbifold [14, 15].

We start in section 2 by reviewing generalities on colliding plane waves in four dimensions, as well as several exact solutions which we will aim at generalizing. In section 3, we propose an ansatz (eq. (3.2) below) for $2n+2$-dimensional CPW solutions, which leads to same Einstein equations as in four dimensions, albeit different boundary conditions. Using this isomorphism, we construct some explicit solutions in $2n+2$ dimensions by upgrading four-dimensional solutions. In section 4, we develop a perturbative scheme which allows us to determine the solution in the vicinity of the light-cone for arbitrary boundary conditions. We use it to study the stability of gravitational and electromagnetic plane waves to small counter-propagating perturbations. Finally, we apply our method to construct higher-dimensional analogues of the Khan-Penrose and Bell-Szekeres metrics.[1] We close in section 5 with a discussion of our results. The results of the perturbative computations can be found in appendix A and B.

[1] Another analogue of the Bell-Szekeres metric was constructed recently, but in the context of higher dimensional Einstein-Maxwell gravity [11].

2. Colliding plane waves in four dimensions

2.1 Generalities

Colliding gravitational plane waves in four dimensions have been an field of intensive study. The metric ansatz compatible with the existence of two commuting spacelike Killing vectors ∂_x, ∂_y is given by the Rosen-Szekeres line element,

$$ds^2 = 2e^{-M}dudv + e^{-U}\left(e^V \cosh W dx^2 - 2\sinh W dxdy + e^{-V}\cosh W dy^2\right), \qquad (2.1)$$

where M, U, V, W are functions of the light-cone coordinates u, v only. We will restrict our attention to colliding waves with aligned polarization, ie $W = 0$. In addition, one may allow for an electromagnetic field,

$$F = \frac{1}{2}\left(dH_1 \wedge dx + dH_2 \wedge dy\right) \qquad (2.2)$$

where H_1 and H_2 are functions of u, v only.

In studying plane wave scattering in flat Minkowski space, one usually assumes that space is flat ahead of each of the incoming plane fronts, say at $u < 0$ and $v < 0$. Space-time is thus divided into four sectors: in the past region $P : u < 0, v < 0$, we have flat Minkowski space with $M = U = V = W = H_i = 0$; the right region $R : u > 0, v < 0$ corresponds to the incoming left-moving plane wave, described by u-dependent profiles $\mathrm{U}(u), V(u)$, etc; similarly, the left region $L : u < 0, v > 0$ corresponds to the right-moving plane wave, described by v-dependent profiles $\mathrm{U}(v), V(v)$, etc. In the forward region $F : u > 0, v > 0$, the two waves start to interact, leading to a metric (2.1) depending non-trivially on both of the light-cone coordinates. The problem is thus to determine the functions U, V, W, M, H_i in the forward region, given their values on the characteristics $u = 0$ and $v = 0$.

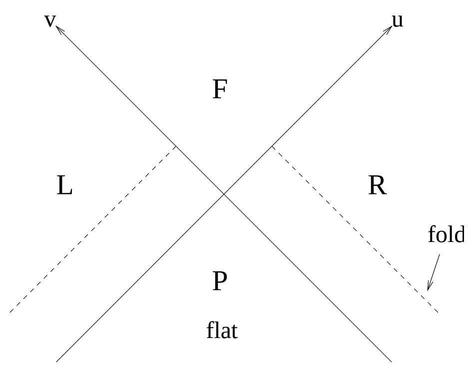

Figure 1: Kinematical set-up for colliding plane waves. The dotted line denote a “fold” singularity in the incoming regions.

For this, notice that a change of u and v coordinates allows to set $M = 0$ in the right (resp. left) region. The other functions U, V, W, H_i in the incoming region are freely chosen functions of u (resp. v), subject to the condition that the corresponding plane wave should satisfy the Einstein equation $R_{uu} = 0$ (resp. $R_{vv} = 0$). In the interacting region, the Einstein equations require that e^{-U} is a free two-dimensional field: its value throughout region F is therefore determined immediately in terms of its boundary values at $u = 0$ and $v = 0$:

$$U = -\log\left[f(u) + g(v)\right] \qquad (2.3)$$

There is no loss of generality in assuming that $f(0) = g(0) = 1/2$. It is often useful then to change coordinates from (u, v) to (f, g). Solving for the other functions V, W, H_i is in general a complicated non-linear problem, except for purely gravitational collinear waves ($W = H_i = 0$), where V satisfies a linear Euler-Darboux equation, and can be determined by Green's function techniques [18, 24]. When either W or H_i are non-zero, the problem is more difficult, although integrability provides solution-generating techniques which in some cases allow to obtain exact solutions. Finally, M may be integrated by quadrature as it satisfies the free 2-dimensional Klein-Gordon equation with sources (Einstein equations for a higher-dimensional generalization of the ansatz (2.1) will be displayed in section 3, (3.4)–(3.8)).

While physically realistic waves have a smooth wave front, it is often useful to allow for mild singularities at $u = 0$ or $v = 0$, in order to describe idealized impulsive or shock wave profiles. As shown by O'Brien and Synge [16], the correct matching conditions to impose at $u = 0$ are that the transverse metric g_{ij} and its derivative $\partial_u g_{ij}$ be continuous. In terms of the ansatz (2.1), this implies that M and V have to be continuous and $f(u)$ and $g(v)$ in (2.3) are at least C^1 across the boundaries $u = 0$ and $v = 0$. In general then,

$$f(u) - \frac{1}{2} \sim u^\alpha \theta(u)\,, \qquad g(v) - \frac{1}{2} \sim -v^\beta \theta(v)\,, \tag{2.4}$$

where $\theta(x)$ is the Heaviside step-function and $\alpha, \beta \geq 2$. For $\alpha = 2$, there are delta functions in components of the curvature tensors which have to be interpreted as distributions [17]. Upon writing the incoming wave in Brinkmann coordinates,

$$ds^2 = 2dx^+dx^- + (H_x(x^+)X_i^2 + H_y(x^+)Y_i^2)(dx^+)^2 + dX_i^2 + dY_i^2\,, \tag{2.5}$$

it is easy to see that this corresponds to an impulsive plane wave, with $H_{x,y} \propto \delta(x^+)$. The $\alpha = 4$ case on the other hand corresponds to a shock wave, with $H_{x,y} \propto \theta(x^+)$.

2.2 Exact four-dimensional CPW solutions

Using the framework just outlined, many exact CPW solutions in four dimensions have been constructed over the years. We now briefly review several interesting solutions that we will be interested in generalizing to higher dimensions, see [2] for an exhaustive review.

A two-parameter family of purely gravitational CPW solutions is given by the Szekeres solution [18]

$$\begin{aligned}
V &= -2k_1 \operatorname{arctanh}\left[\sqrt{\frac{\frac{1}{2}-f}{\frac{1}{2}+g}}\right] - 2k_2 \operatorname{arctanh}\left[\sqrt{\frac{\frac{1}{2}-g}{\frac{1}{2}+f}}\right], && (2.6)\\
W &= H = 0\,, && (2.7)\\
M &= +\frac{1}{2}(1-(k_1+k_2)^2)\ln(f+g) + \frac{k_2^2}{2}\ln\left(\frac{1}{2}+f\right) + \frac{k_1^2}{2}\ln\left(\frac{1}{2}+g\right) - \\
&\quad -2k_1k_2 \ln\left(\sqrt{\frac{1}{2}-f}\sqrt{\frac{1}{2}-g} + \sqrt{\frac{1}{2}+f}\sqrt{\frac{1}{2}+g}\right), && (2.8)
\end{aligned}$$

where $f = \frac{1}{2} - u^\alpha$ and $g = \frac{1}{2} - v^\beta$ as in (2.4) and $k_1^2 = 2(\alpha-1)/\alpha$, $k_2^2 = 2(\beta-1)/\beta$. A case of particular interest is the Khan-Penrose solution [1], which arises for $\alpha = \beta = 2$. In the incoming region R, it corresponds to a profile

$$ds^2|_R = 2dudv + [1 + u\theta(u)]^2 dx^2 + [1 - u\theta(u)]^2 dy^2 \tag{2.9}$$

which reduces to flat space both before ($u < 0$) and after ($u > 0$) the wave front. The singularity at $u = 0$ describes an impulsive gravitational wave, with delta function profile. The singularity at $u = -1$ on the other hand is merely a coordinate singularity, often called "fold singularity" in the literature. The profile in the region L is identical to (2.9) up to exchanging u and v. In the interacting region however, the geometry is curved, with a space-like curvature singularity at $u^2 + v^2 = 1$. The metric furthermore becomes complex at $u > 1$ or $v > 1$. It is thus legitimate to excise the region behind the space-like singularity in region F, as well as behind the fold singularity in regions L, R. Indeed all but exceptional $u = cste$ causal geodesics coming from region P fall into the singularity in F [19].

Another interesting explicit solution is the Bell-Szekeres solution [20], which describes colliding electromagnetic waves in Einstein-Maxwell gravity. The solution in the interaction region takes the very simple form given by

$$ds^2 = 2dudv + \cos^2(u-v)dx^2 + \cos^2(u+v)dy^2 \,, \tag{2.10}$$

$$A_y = \frac{1}{\sqrt{2k}} \sin(u+v) \tag{2.11}$$

where the replacements $u \to u\theta(u)$ and $v \to v\theta(v)$ are implicit. In region R, the incoming wave may be written in Brinkmann coordinates,

$$ds^2 = 2dX^+dX^- + dX^2 + dY^2 - \frac{1}{4}(X^2+Y^2)(dX^+)^2 \tag{2.12}$$

which we recognize the standard electromagnetic plane wave, One of the most interesting properties of the Bell-Szekeres solution is that the metric (2.10) in the interaction region F is in fact diffeomorphic to a slice of the Bertotti-Robinson $AdS_2 \times S^2$ space [21, 22]. In contrast of the purely gravitational collision described above, the collision of two collinear elecromagnetic waves is therefore free of any singularity. Instead, the Killing vectors ∂_x and ∂_y become null at $u + v = (2k+1)\pi/2$ and $u - v = (2l+1)\pi/2$, respectively. There is therefore a Killing horizon at $u + v = \pi/2$, joining on to the fold singularities at $u = \pi/2$ and $v = \pi/2$ in the incoming regions. In the literature, one usually excises the region behind the fold singularity and the Killing horizon, although this appears to be much less justified than in the Khan-Penrose- Szekeres case.

Note that the incoming plane wave (2.12) is the four dimensional analogue of the ten dimensional type-IIB plane wave found recently by Blau - Figueroa o'Farrill - Hull - Papadopoulos (BFHP) [12]. One of the goals of this note is to analyze the scattering of such PP-waves in higher dimensions, a situation which is of interest in string theory. In order to avoid undue suspense, let us immediately state that in contrast to the Bertotti-Robinson space, $AdS_5 \times S^5$ cannot be viewed as the collision of two BFHP plane waves.

Finally, we would like to stress that there are two ingredients in the dynamics of colliding plane waves: firstly the equations of motions and secondly the matching of the solution in the interaction region F to incoming plane waves in region L and R and flat space in region P. Different matching prescriptions define different (un)physical problems. In particular, Gowdy-type cosmologies satisfy the same equations of motion (usually displayed using different coordinates) but different boundary conditions.

3. Colliding plane waves in arbitrary dimensions

Having reviewed the basic features and simplest CPW solutions of Einstein-Maxwell gravity in four dimensions, we now generalize these results to gravity in even dimension $D = 2n+2$ with a minimally coupled $n+1$ form field strength, whose action is given by

$$S = \int d^D x \sqrt{-g} \left(R - \frac{k}{2(n+1)!} F_{\mu_1 \cdots \mu_{n+1}} F^{\mu_1 \cdots \mu_{n+1}} \right) . \tag{3.1}$$

Our main interest actually lies in the $n = 4$ case, which describes a subsector of ten-dimensional type-IIB supergravity, upon restricting to self-dual configurations of the 5-form field strength. Other cases $n = 2, 3$ may also be relevant in the context of type-IIB string theory and F-theory compactified on K_3, respectively. This motivates the following ansätze for the metric,

$$ds^2 = 2e^{-M} du dv + e^{-\frac{1}{n}(U-V)}(dx_1^2 + \cdots + dx_n^2) + e^{-\frac{1}{n}(U+V)}(dy_1^2 + \cdots + dy_n^2) . \tag{3.2}$$

and for the $n+1$-index anti-symmetric tensor field strength,

$$\begin{aligned} F_{n+1} = \frac{1}{\sqrt{n}} \Big(& \partial_u H_1 du \wedge dx_1 \wedge \cdots \wedge dx_n + \partial_v H_1 dv \wedge dx_1 \wedge \cdots \wedge dx_n + \\ & + \partial_u H_2 du \wedge dy_1 \wedge \cdots \wedge dy_n + \partial_v H_2 dv \wedge dy_1 \wedge \cdots \wedge dy_n \Big) . \end{aligned} \tag{3.3}$$

Indeed, these ansätze are appropriate for incoming BFHP waves in type-IIB supergravity [26] (or their analogues in lower dimension), which preserve an $\mathrm{SO}(n) \times \mathrm{SO}(n)$ symmetry. Here M, U, V, H_1 and H_2 are all functions of u and v only. Factors of n have been inserted for later convenience. One may also consider non-collinear polarization, or switch on other fields present in supergravity such as the dilaton, axion or other antisymmetric tensor fields, but we shall refrain from doing so. Our excuse is that, besides simplicity, the equations of motion satisfied by the ansatz (3.2)–(3.3) are in fact identical to those satisfied by the four-dimensional Rosen-Szekeres solutions (2.1)–(2.2).[2]

3.1 Equations of motion

Indeed, just as in four dimensions [18, 2], the Einstein equations on the ansatz (3.2) can be written as two chiral equations,

$$\partial_u^2 U + \partial_u U \partial_u M - \frac{1}{2n} \left((\partial_u U)^2 + (\partial_u V)^2 \right) = \frac{k}{2n} \left(e^{U-V} (\partial_u H_1)^2 + e^{U+V} (\partial_u H_2)^2 \right) \tag{3.4}$$

$$\partial_v^2 U + \partial_v U \partial_v M - \frac{1}{2n} \left((\partial_v U)^2 + (\partial_v V)^2 \right) = \frac{k}{2n} \left(e^{U-V} (\partial_v H_1)^2 + e^{U+V} (\partial_v H_2)^2 \right) \tag{3.5}$$

and three integrability conditions,

$$\partial_u \partial_v U - \partial_u U \partial_v U = 0 , \tag{3.6}$$

$$2\partial_u \partial_v V - \partial_u V \partial_v U - \partial_v V \partial_u U + 2k \left(e^{U-V} \partial_u H_1 \partial_v H_1 - e^{U+V} \partial_u H_2 \partial_v H_2 \right) = 0 , \tag{3.7}$$

$$\partial_u \partial_v M - \frac{1}{2n} \partial_u V \partial_v V + \frac{2n-1}{2n} \partial_u U \partial_v U = 0 . \tag{3.8}$$

[2] This would remain true if we allowed non-parallel polarizations preserving the $\mathrm{SO}(n) \times \mathrm{SO}(n)$ symmetry.

Satisfying these equations automatically imply (3.4) and (3.5). In the L and R regions, the latter are simply the equations $R_{uu}=0$ and $R_{vv}=0$ satisfied by the incoming plane waves. Note that the constraint (3.4) (resp. 3.5) implies that the mean radius of the transverse directions $e^{-U/2n}$ is a concave function of u (resp. v) except when it vanishes. $f(u)$ (resp. $g(v)$) is therefore a concave decreasing function, indicating that a (possibly coordinate) singularity in regions L amd R is inevitable. Thus "fold" singularities are generic in higher dimensions as well. Note also that these equations automatically imply the vanishing of the Ricci scalar

$$R=-\frac{1}{n}e^{M}\left[(2n+1)\partial_u U\partial_v U+\partial_u V\partial_v V-2n(\partial_u\partial_v M+2\partial_u\partial_v V)\right]. \tag{3.9}$$

Other curvature invariants $R_2=(R_{\mu\nu})^2$ and $R_4=(R_{\mu\nu\rho\sigma})^2$ are however non-trivial, and as we shall see generically have singularities in the interacting region. Finally, it is easy to check that that the only conformally flat plane-wave solution satisfying the ansatz (3.2)–(3.3) is flat $d+2$-dimensional Minkowski space, with the sole exception of the Bertotti-Robinson metric $AdS_2\times S^2$ in 4 dimensions. In particular, $AdS_5\times S^5$ does not fit into the ansatz (3.2)–(3.3)in 10 dimensions.

In addition to the Einstein equations (3.4)–(3.8), the equations of motion for the n-form electromagnetic gauge field are given by

$$2\partial_u\partial_v H_1-\partial_u H_1\partial_v V-\partial_v H_1\partial_u V=0\,, \tag{3.10}$$

$$2\partial_u\partial_v H_2+\partial_u H_2\partial_v V+\partial_v H_2\partial_u V=0\,. \tag{3.11}$$

In dimension $d=6,10$ one can impose a self-duality of the fieldstrength $F=*F$ which relates the two functions H_1 and H_2 by

$$\partial_u H_2=e^{-V}\partial_u H_1\,,\qquad \partial_v H_2=-e^{-V}\partial_v H_1\,. \tag{3.12}$$

This is notably the case in type-IIB supergravity. In dimensions $d=4,8$ there exist no self-duality condition in lorentzian signature, nevertheless (3.12) may still be imposed. For simplicity, we shall impose (3.12) for all values of n, and denote $H=H_1$. The constraints (3.4)–(3.5) thus reduce to

$$\partial^2 U+\partial U\partial M-\frac{1}{2n}\Big((\partial U)^2+(\partial V)^2\Big)=\frac{k}{n}e^{U-V}(\partial H)^2\,, \tag{3.13}$$

where ∂ stands either for ∂_u or ∂_v.

As in four dimensions, (3.6) implies that $e^{-\mathrm{U}(u,v)}$ is a free field in two dimensions, and therefore

$$U=-\log\big(f(u)+g(v)\big)\,. \tag{3.14}$$

The functions $f(u)$ and $g(v)$ can be directly determined from the value of U on the characteristics $u=0$ and $v=0$, assuming $f(0)=g(0)=1/2$. For purely gravitational waves ($H=0$), eq. (3.7) is the linear Euler-Darboux equation, and can be solved by the same Green's function techniques that applied in the four-dimensional case. When $H\neq 0$, the equations (3.7) and (3.10) are non-linear, but identical to the four-dimensional case: they can thus be dealt with just the same integrability techniques. The constraints (3.4)-(3.5) can then be integrated to yield the remaining function M. Changing coordinates from u, v to $f(u), g(v)$, they reduce to

$$\partial_f M + \frac{f+g}{2n}(\partial_f V)^2 + \frac{k}{n}e^{-V}(\partial_f H)^2 - \frac{2n-1}{2n}\frac{1}{f+g} + \frac{\partial_u^2 f}{(\partial_u f)^2} = 0\,, \tag{3.15}$$

$$\partial_g M + \frac{f+g}{2n}(\partial_g V)^2 + \frac{k}{n}e^{-V}(\partial_g H)^2 - \frac{2n-1}{2n}\frac{1}{(f+g)} + \frac{\partial_v^2 g}{(\partial_v g)^2} = 0\,. \tag{3.16}$$

As in 4 dimensions, the inhomogeneous terms may be absorbed by defining

$$M(u,v) = \frac{2n-1}{2n}\log(f+g) - \log(\partial_u f) - \log(\partial_v g) + S(f(u), g(v))\,. \tag{3.17}$$

The constraint equations then simplify to

$$\partial_f S + \frac{1}{n}\Big(k\, e^{-V}(\partial_f H)^2 + \frac{f+g}{2}(\partial_f V)^2\Big) = 0\,, \tag{3.18}$$

$$\partial_g S + \frac{1}{n}\Big(k\, e^{-V}(\partial_g H)^2 + \frac{f+g}{2}(\partial_g V)^2\Big) = 0\,, \tag{3.19}$$

involving derivatives with respect to f, g only.

This shows that the equations of motion for the $2n+2$-dimensional ansatz (3.2)–(3.3) with "self-dual" $n+1$ field-strength are isomorphic to those for the $n = 1$ four-dimensional collinear ansatz (2.1)–(2.2), upon identifying[3]

$$U_n = U_1\,, \qquad V_n = V_1\,, \qquad H_n = H_1\,, \tag{3.20}$$

$$M_n = \frac{1}{n}M_1 + \frac{n-1}{n}\big(\log(f+g) - \log(\partial_u f) - \log(\partial_v g)\big)\,. \tag{3.21}$$

This immediately allows us to construct $2n+2$-dimensional solutions from four dimensional ones, as we now discuss.

3.2 Higher dimensional CPW from four-dimensions

As discussed in the previous section the form of the solution has a very simple dependence on n. In particular in the "self-dual" case the equations for U and V are independent of n. This implies that a CPW wave solution in $d = 4$ ($n = 1$) automatically gives a solution in $d = 2n + 2$, with $n > 1$. A word of caution however is that, due to the non-homogeneous terms $\log(\partial_u f)$, $\log(\partial_v g)$ in (3.21), the properties of the resulting solution can change significantly for $n > 1$. In particular the existence of a smooth matching of the solution for $n = 1$ does not in general imply a smooth solution in higher dimensions. In the following we will analyze the properties of higher dimensional solution for some particular four dimensional solutions.

[3]This isomorphism remains valid upon dropping the assumptions of collinearity and self-duality.

Purely gravitational higher dimensional CPW. To illustrate this construction, we start with the four-dimensional Szekeres solution (2.8) with

$$f(u) = \frac{1}{2} - u^{\alpha}\theta(u)\,, \qquad g(v) = \frac{1}{2} - v^{\beta}\theta(v)\,. \tag{3.22}$$

Using the upgrading procedure outlined above, one finds a four-parameter family of purely gravitational CPW solutions in arbitrary dimension $2n+2$

$$U = -\ln(1-u^{\alpha}-v^{\beta})\,, \tag{3.23}$$

$$V = -2k_1 \operatorname{arctanh}\left[\sqrt{\frac{u^{\alpha}}{1-v^{\beta}}}\right] - 2k_2 \operatorname{arctanh}\left[\sqrt{\frac{v^{\beta}}{1-u^{\alpha}}}\right], \tag{3.24}$$

$$\begin{aligned} M =& -\ln\partial_u f + \frac{k_1^2}{2n}\log\left(\frac{1}{2}-f\right) - \ln\partial_v g + \frac{k_2^2}{2n}\log\left(\frac{1}{2}-g\right) + \\ &+ \frac{2n-1-(k_1+k_2)^2}{2n}\ln(1-u^{\alpha}-v^{\beta}) + \frac{k_2^2}{2n}\ln(1-u^{\alpha}) + \frac{k_1^2}{2n}\ln(1-v^{\beta}) + \\ &+ \frac{k_1k_2}{n}\ln\left(1-u^{\alpha}-v^{\beta}+2u^{\alpha}v^{\beta}+2\sqrt{u^{\alpha}v^{\beta}(1-u^{\alpha})(1-v^{\beta})}\right). \end{aligned} \tag{3.25}$$

where we implicitely replace $u^{\alpha} \to u^{\alpha}\theta(u)$ and $v^{\beta} \to v^{\beta}\theta(v)$. As in four-dimensions however, the two parameters k_1, k_2 get related to α, β by demanding that M be continuous across $u=0$ and $v=0$,

$$k_1^2 = 2n\frac{\alpha-1}{\alpha}\,, \qquad k_2^2 = 2n,\frac{\beta-1}{\beta}\,. \tag{3.26}$$

The family of physical solutions relevant in dimension $2n+2$ is therefore different from that relevant in dimension 4. The fact that a curvature singularity arises at $u\alpha + u^{\beta} = 1$ remains nevertheless true. The solution satisfies the appropriate junction conditions across the null surface $u=0$ (resp. $v=0$) if $\alpha > 2$ (resp. $\beta > 2$). In region R, the incoming plane wave metric is given by

$$\begin{aligned} ds^2 =& \big(1-u^{\alpha}\theta(u)\big)^{-\frac{2n-\alpha}{2n\alpha}}\, 2dudv + \big(1-u^{\alpha}\theta(u)\big)^{\frac{1}{n}}\left(\frac{1+u^{\frac{\alpha}{2}}\theta(u)}{1-u^{\frac{\alpha}{2}}\theta(u)}\right)^{\sqrt{\frac{2(\alpha-1)}{n\alpha}}} dx_i^2 + \\ &+ \big(1-u^{\alpha}\theta(u)\big)^{\frac{1}{n}}\left(\frac{1+u^{\frac{\alpha}{2}}\theta(u)}{1-u^{\frac{\alpha}{2}}\theta(u)}\right)^{-\sqrt{\frac{2(\alpha-1)}{n\alpha}}} dy_i^2\,. \end{aligned} \tag{3.27}$$

In order to transform this metric in Brinkmann coordinates one has to redefine the u coordinate for $u>0$

$$\big(1-u^{\alpha}\theta(u)\big)^{-\frac{2n-\alpha}{2n\alpha}} du = d\tilde{u}\,. \tag{3.28}$$

Then the metric can be brought into the form

$$ds^2 = 2d\tilde{u}dv + e_x(\tilde{u})^2 dx_i^2 + e_y^2(\tilde{u})dy_i^2 \tag{3.29}$$

and the standard change of variables into Brinkmann form gives

$$ds^2 = 2dx^+dx^- + (H_x(x^+)X_i^2 + H_y(x^+)Y_i^2)(dx^+)^2 + dX_i^2 + dY_i^2, \tag{3.30}$$

where

$$H_x = \frac{1}{e_x}\frac{d^2 e_x}{d\tilde{u}^2}\,, \qquad H_y = \frac{1}{e_y}\frac{d^2 e_y}{d\tilde{u}^2}\,. \tag{3.31}$$

An analogue of the four-dimensional Khan-Penrose solution may be obtained by setting $\alpha = \beta = 2$, i.e. $k_1^2 = k_2^2 n$: the resulting incoming profile

$$H_x = \frac{1}{\sqrt{n}}\delta(x^+) + \frac{n-1}{n^{\frac{3}{2}}}\frac{x^+}{(1-(x^+)^2)^{\frac{n+1}{n}}}\theta(x^+)\,, \tag{3.32}$$

$$H_y = -\frac{1}{\sqrt{n}}\delta(x^+) - \frac{n-1}{n^{\frac{3}{2}}}\frac{x^+}{(1-(x^+)^2)^{\frac{n+1}{n}}}\theta(x^+)\,. \tag{3.33}$$

shows an impulsive (delta function) component,together with a non-vanishing tail which depends on x^+: this is markedly different from 4-dimensional case, where space was flat on either side of the wave front. We shall return to the true analogue of the Khan-Penrose solution in sections 3.3 and 4.2.

Similarly, a higher dimensional analog of the Szekeres solution is obtained by $k_1 = k2 = \sqrt{2n\frac{m-1}{m}}$ which gives $\alpha = \beta = m$ with integer $m \geq 3$ and one finds

$$H_x = h_x(x^+)\theta(x^+)\,, \qquad H_y = h_y(x^+)\theta(x^+)\,, \tag{3.34}$$

where the behavior of h_x, h_y as $x^+ \to 0$ depends on the value of the integer m. For $m = 3$, h_x, h_y diverge as $x^+ \to 0$. For $m = 4$, h_x, h_y has a discontinuity at $x^+ \to 0$, but also exhibit a x^+ dependent tail: this is in contrast to the 4-dimensional Szekeres solution, which corresponded to a true shock wave with constant Brinkmann parameter on either side of the wave front. For $m > 4$ the functions h_x, h_y vanish as $x^+ \to 0$.

To summarize, by upgrading the general Szekeres solution we found exact purely gravitational $\mathrm{SO}(n) \times \mathrm{SO}(n)$ symmetric collinear CPW in higher dimensions. They describe different incoming profiles from the 4-dimensional case, yet exhibit the same features of fold singularities in the R, L region and space-like singularity in the forward F region.

Electromagnetic CPW solutions. We now upgrade the four dimensional Bell-Szekeres solution (2.10)–(2.11) into a higher dimensional colliding plane wave solution with flux. Solving (3.4) and (3.5) determines M to be

$$M = \frac{n-1}{n}\log\left(\frac{\cos(u-v)\cos(u+v)}{\sin(2u)\sin(2v)}\right), \tag{3.35}$$

$$H = = \frac{1}{\sqrt{2k}}\sin(u+v) \tag{3.36}$$

hence the metric is of the following form

$$ds = 2\left(\frac{\sin(2u)\sin(2v)}{\cos(u-v)\cos(u+v)}\right)^{\frac{n-1}{n}} dudv + \cos^{\frac{2}{n}}(u+v)(dx_1^2 + \dots) + \cos^{\frac{2}{n}}(u-v)(dy_1^2 + \cdots)\,. \tag{3.37}$$

Unfortunately, this metric exhibit a curvature singularity on the characteristics $u = 0$ and $v = 0$, as can be seen by computing the scalar curvature invariant

$$R_2 = \left(\frac{\cos(u-v)\cos(u+v)}{\sin(2u)\sin(2v)} \right)^{\frac{2(n-1)}{n}} . \tag{3.38}$$

This implies that the solution in the interaction region cannot be glued smoothly to incoming plane waves in region L and R, and is therefore unphysical. Another way to reach the same conclusion is to change variables to $\tilde{u} = (\sin u)^{\frac{2n-1}{n}}, \tilde{v} = (\sin v)^{\frac{2n-1}{n}}$, so as to remove the singular part of the $dudv$ term in (3.37): then $f = 1/2 - \hat{u}^{\frac{2n}{2n-1}}$, $g = 1/2 - \hat{v}^{\frac{2n}{2n-1}}$ have exponents α, β smaller than two, in contradiction with the O'Brien Synge conditions. Finally, note that the incoming plane waves in the L and R regions are markedly different from the BFHP plane wave in higher dimensions, which was our motivation to look at the Bell-Szekeres solution in the first place.

While this attempt to produce a physical higher-dimensional CPW solution from the Bell-Szekeres solution has failed, there are nevertheless four-dimensional electromagnetic CPW solutions which lead to acceptable higher-dimensional ones. For example, recall that using solution-generating techniques developped in four dimensions, a purely gravitational CPW solution U_0, V_0, M_0 may be turned into an electromagnetic one by the following transformation:

$$\begin{aligned} U &= U_0 , \qquad V = V_0 - 2\ln\left(\cos^2\alpha + \sin^2\alpha\, e^{-U_0+V_0}\right) \\ M &= M_0 - 2\ln\left(\cos^2\alpha + \sin^2\alpha\, e^{-U_0+V_0}\right), \qquad H = \sqrt{\frac{2}{k}} \frac{\sin\alpha\cos\alpha\,(e^{-U_0+V_0}-1)}{\cos^2\alpha + \sin^2\alpha\, e^{-U_0+V_0}} \end{aligned} \tag{3.39}$$

Starting from the four-dimensional purely gravitational Szekeres solution (2.8), one may thus obtain a 5-parameter family of higher dimensional CPW solutions with flux. A special choice of parameters then leads to a solution which is regular at the characteristics $u = 0$ and $v = 0$, although it displays a curvature singularity at $f(u) + g(v) = 0$ in the interaction region.

3.3 Toward higher-dimensional Khan-Penrose and Bell-Szekeres CPW

The construction in the previous section, based on upgrading known four-dimensional solutions into higher dimensional ones, has failed to produce what deserves to be called higher-dimensional analogs of the Khan-Penrose and Bell-Szekeres solutions, namely solutions describing the collision of two purely gravitational impulsive plane waves, and two purely electromagnetic shock waves, respectively. This must simply mean that we failed to impose the appropriate boundary conditions on the characteristics.

Indeed, a higher-dimensional analog of the Khan-Penrose solution should describe the collision of two plane waves with Rosen coordinate metric

$$ds^2 = 2dudv + [1 + u\theta(u)]^2(dx_1^2 + \cdots + dx_n^2) + [1 - u\theta(u)]^2(dy_1^2 + \cdots + dy_n^2) , \tag{3.40}$$

flat on either side of the wave front at $u = 0$. The appropriate boundary conditions are therefore

$$\begin{aligned}
\mathrm{U}(u,v) &= -\log[(1-u^2)^n + (1-v^2)^n - 1] && (3.41)\\
V(u,0) &= n\log\left[\frac{(1+u)}{(1-u)}\right] && (3.42)\\
V(0,v) &= n\log\left[\frac{(1+v)}{(1-v)}\right]. && (3.43)
\end{aligned}$$

A solution could be obtained using the Green function technique developed in [18, 24], however for $n > 1$ the resulting integrals appear to be too difficult.

Similarly, a higher-dimensional analog of the Bell-Szekeres solution should describe the collision of two BFHP plane waves with Rosen coordinate metric and F field

$$\begin{aligned}
ds^2 &= 2dudv + \cos^2 u\ (dx_1^2 + \cdots + dx_n^2 + dy_1^2 + \cdots + dy_n^2) && (3.44)\\
H &= 2\int_0^u \cos^{2n} u du\,. && (3.45)
\end{aligned}$$

The appropriate boundary conditions are therefore

$$\begin{aligned}
\mathrm{U}(u,v) &= -\log[\cos^{2n} u + \cos^{2n} v - 1] && (3.46)\\
V(u,0) &= 0\,, \qquad V(0,v) = 0 && (3.47)\\
H(u,0) &= 2\int_0^u \cos^{2n} u du\,, \qquad H(0,v) = 2\int_0^v \cos^{2n} v dv\,. && (3.48)
\end{aligned}$$

Again, it is not clear how to obtain such a solution with the known solution-generating techniques in four-dimensions. In the following we will follow a different approach, using perturbation theory around the light-cone.

4. Perturbative plane wave collisions

In this section, our aim is to analyze higher dimensional plane wave collisions in a perturbative expansion around the light-cone — or, equivalently, when one of the waves is of much smaller amplitude than the other. As an application, we shall obtain approximations to the higher-dimensional Khan-Penrose and Bell-Szekeres CPW solutions as defined in section 3.3, in the vicinity of the light-cone. We will also be able to study generic perturbations of the impulsive and shock plane waves in higher dimensions.

4.1 General set-up

We consider an incoming left-moving plane wave in $2n+2$ dimensions, given by the Rosen coordinate metric and flux

$$\begin{aligned}
ds^2|_R &= 2dudv + e^{-\frac{1}{n}U^{(0)}(u)}\left[e^{\frac{1}{n}V^{(0)}(u)}(dx_1^2 + \cdots + dx_n^2) + e^{-\frac{1}{n}V^{(0)}(u)}(dy_1^2 + \cdots + dy_n^2)\right]\\
F_{n+1} &= \frac{1}{2}\partial_u H^{(0)}(u)\ du \wedge (dx_{1\ldots n} + dy_{1\ldots n})\,. && (4.1)
\end{aligned}$$

Here we took advantage of coordinate reparametrization invariance to set $M = 0$, and restricted to self-dual flux configurations. The functions $U^{(0)}, V^{(0)}, H^{(0)}$ are assumed to satisfy the chiral equation of motion (3.4). We also assume that these functions vanish at $u < 0$, and satisfy the appropriate regularity conditions at $u = 0$.

On the other hand, the incoming right-moving plane wave is assumed to be of the form

$$\begin{aligned} ds^2|_L &= 2dudv + e^{-\frac{1}{n}U^{(1)}(\epsilon v)} \left[e^{\frac{1}{n}V^{(1)}(\epsilon v)}(dx_1^2 + \cdots + dx_n^2) + e^{-\frac{1}{n}V^{(1)}(\epsilon v)}(dy_1^2 + \cdots + dy_n^2) \right] \\ F_{n+1} &= \frac{1}{2}\partial_u H^{(1)}(\epsilon v)\; du \wedge (dx_{1\ldots n} + dy_{1\ldots n}) \end{aligned} \tag{4.2}$$

where we introduced a parameter ϵ which can be viewed as the relative strength of the two plane waves: upon going to Brinkmann coordinates, the Brinkmann mass parameter is proportional to ϵ. As for the right-moving wave, we assume that $U^{(1)}, V^{(1)}, H^{(1)}$ vanish at $v < 0$, and have at most an impulsive singularity at $v = 0$. For simplicity, we also restrict to collinear polarization.

We shall be interested in a perturbative expansion in ϵ, therefore in the vicinity of the characteristic axis $v = 0$ in the forward region F. To this purpose, we expand the left-moving wave profile as

$$g^{(1)}(v > 0) = \frac{1}{2} + \frac{1}{2!}g_2 v^2 + \frac{1}{3!}g_3 v^3 + \cdots \tag{4.3}$$

$$V^{(1)}(v > 0) = v_1 v + \frac{1}{2!}v_2 v^2 + \frac{1}{3!}v_3 v^3 + \cdots \tag{4.4}$$

$$H^{(1)}(v > 0) = h_1 v + \frac{1}{2!}h_2 v^2 + \frac{1}{3!}h_3 v^3 + \cdots \tag{4.5}$$

where $g^{(1)}(v)$ is defined as usual by $U^{(1)}(v) = -\log[g^{(1)}(v) + 1/2]$. The vanishing of g_1, v_0, h_0 follows from our assumptions on the singularity at $v = 0$. The chiral equation (3.5) allows to eliminate e.g. $g^{(1)}$ in favor of $V^{(1)}, H^{(1)}$ order by order in v, e.g.

$$g_2 = -\frac{1}{2n}\left(h_1^2 + {v_1}^2\right), \qquad g_3 = \frac{1}{2n}\left(2h_1h_2 - h_1^2 v_1 + 2v_1v_2\right) \tag{4.6}$$

$$\begin{aligned} g_4 = -\frac{1}{4n^3}\Big[& h_1^4\,(1-2n) + {v_1}^4 + 4h_1n^2\,(h_3 - 2h_2v_1) + h_1^2\left((n-2)\,(2n-1)\,{v_1}^2 - 2n^2v_2\right) + \\ & + n\left(-3{v_1}^4 + 4n\left({h_2}^2 + {v_2}^2 + v_1v_3\right)\right)\Big]\,. \end{aligned} \tag{4.7}$$

Our goal is now to solve for the solution in the interacting region. Using the field equation (3.4), U is determined throughout the interacting region by its value on the characteristics $uv = 0$:

$$\mathrm{U}(u, v) = -\log\left[f^{(0)} + g^{(1)}(\epsilon v)\right] \tag{4.8}$$

where we defined as usual $U^{(0)}(u) = -\log[f^{(0)}(u) + 1/2]$. The other functions V, H, M in the ansatz (3.2)–(3.3) can be determined order by order in ϵ by Taylor expanding

$$\begin{aligned} H &= H^{(0)}(u) + \epsilon v H_1(u) + \frac{1}{2!}\epsilon^2 v^2 H_2(u) + \frac{1}{3!}\epsilon^2 v^2 H_3(u) + \cdots \\ V &= 0 + \epsilon v V_1(u) + \frac{1}{2!}\epsilon^2 v^2 V_2(u) + \frac{1}{3!}\epsilon^2 v^3 V_3(u) + \cdots \\ M &= 0 + \epsilon v M_1(u) + \frac{1}{2!}\epsilon^2 v^2 M_2(u) + \frac{1}{3!}\epsilon^3 v^3 M_3(u) + \cdots\,. \end{aligned} \tag{4.9}$$

The validity of this expansion may be checked by self-consistency, or by looking at the known Khan-Penrose and Bell-Szekeres solution for $n = 1$.

4.2 Higher-dimensional Kahn-Penrose solution

We now specialize to the case where the incoming right-moving plane wave is a purely gravitational impulsive profile (3.40), hence

$$\mathrm{U}(u, v) = -\log\left[(1-u^2)^n + g^{(1)}(\epsilon v) - \frac{1}{2}\right] \tag{4.10}$$

At leading order in ϵ, the equations (3.6)–(3.8) for $H_1(u), V_1(u), M_1(u)$ are first order homogeneous linear ODE's,

$$H_1' - \frac{n}{1-u^2} H_1 = 0 \tag{4.11}$$

$$V_1' - \frac{nu}{1-u^2} V_1 = 0 \tag{4.12}$$

$$M_1' - \frac{1}{1-u^2} V_1 = 0\,. \tag{4.13}$$

Using the boundary conditions $H_1(0) = h_1, V_1(0) = v_1, M_1(0) = 0$, we obtain

$$H_1(u) = h_1\ (1+u)^{n/2}\ (1-u)^{-n/2} \tag{4.14}$$

$$V_1(u) = v_1\ (1-u^2)^{-n/2} \tag{4.15}$$

$$M_1(u) = v_1\ u\ {}_2F_1\left(\frac{1}{2}, 1+\frac{n}{2}, \frac{3}{2}, u^2\right) \tag{4.16}$$

where the hypergeometric function reduces to a simple algebraic function when n is odd. The norm of the Killing vector $\partial_{x,y}$ thus becomes

$$|\partial_{x,y}|^2 = (1-u)^2 \pm \frac{1}{n} v_1 (1-u)^2 (1-u^2)^{-n/2} (\epsilon v) + \mathcal{O}(\epsilon^2) \tag{4.17}$$

which implies that the Killing horizons are shifted to

$$|\partial_{x,y}|^2 = 0: \quad \epsilon v \sim \frac{1}{v_1} n\ 2^{n/2} \Delta^{n/2} \tag{4.18}$$

where $\Delta = 1 - u$. This is in fact an upper estimate, since higher order terms in (4.17), if singular at $u = 1$, may lead to a critical exponent higher than $n/2$. At this order, the curvature invariants R_2 and R_4 remain zero however.

At second order in ϵ, the equations for $H_2(u), V_2(u), M_2(u)$ become first order linear ODE's with a source,

$$H_2' - \frac{n}{1-u^2} H_1 = \frac{1}{2} h_1 v_1 n (1-u)^{-n-1} \tag{4.19}$$

$$V_2' - \frac{nu}{1-u^2} V_1 = \frac{1}{2}\left[v_1^2 - (n+1) h_1^2\right] (1-u^2)^{-n-1} \tag{4.20}$$

$$M_2' - \frac{1}{1-u^2} V_2 = -\frac{1}{2n} u (1-u^2)^{-n-1} \left[(2n-1) h_1^2 + (n-1) v_1^2\right]. \tag{4.21}$$

These equations are too complicated to give a general answer for any n, however explicit solutions can be found in appendix B for the values $n = 1 \dots 4$ of interest. The main result here is that the curvature invariant R_4 is now non-zero[4] and in fact diverges in the interacting region at $u = 1$, as $R_4 \sim \Delta^{n+2}$.

We have computed the solution up to order ϵ^4 for $n = 1 \dots 4$ in the special case where the left-moving (v) profile is purely gravitational, i.e all $h_i = 0$. It is then consistent to set $H(u, v) = 0$ throughout the interaction region. This allows us to obtain the curvature invariant R_4 to second order in ϵ (see appendix A for explicit results). We find in general that higher order corrections diverge faster. Looking at the zeros of $1/R^4$, we find that higher order corrections shift the precise location of the pole. For example, in the $n = 4$ case relevant for type-IIB, we find

$$R_4^{-1} = \frac{16\Delta^6}{3v_1^2} - \frac{11v_1\Delta^4}{6v_1}\epsilon v + \frac{41v_1^2\Delta^2}{192}(\epsilon v)^2 + \mathcal{O}(\epsilon^3) \tag{4.22}$$

which shows that the pole is shifted to[5]

$$R_4 = \infty : \quad \epsilon v = \mathcal{O}(\Delta^{n+2}) \,. \tag{4.23}$$

Determining the precise coefficient would require to resum the series in (4.22). In fact, more divergent terms in higher order contributions to R^4 could even increase the exponent, so that the relation provides only an upper estimate. To fourth order, we find that the critical exponent for the location of the Killing horizon (4.18) remains equal to $n + 2$. In contrast to the $n = 1$ case however, the numerical coefficients appear to be different: it is therefore conceivable that the space-like singularity may be hidden behind a Killing horizon, in agreement with cosmic censorship.

Finally, a case of particular interest is when the two colliding waves have identical impulsive profile: this is the higher dimensional analog of the Khan-Penrose solution. The solution may be obtained to order ϵ^4 by setting

$$v_1 = 2n \,, \qquad v_2 = 0 \,, \qquad v_3 = 4n \,, \qquad v_4 = 0 \,, \tag{4.24}$$

in the formulae in appendix A.

4.3 Higher-dimensional Bell-Szekeres solution

We now consider a purely electromagnetic shock wave as our right-moving background profile given by (3.44). The function U is thus determined throughout the interaction region by

$$\mathrm{U}(u, v) = -\log\left[\cos^{2n} u + g^{(1)}(\epsilon v) - \frac{1}{2}\right] . \tag{4.25}$$

For the most part, we shall consider purely gravitational left-moving perturbations, i.e. $v_i = 0$. We shall however relax this assumption in the $n = 1$ case when discussing the stability of its horizon.

[4] Since they arise only at order ϵ^2, we systematically rescale R_2 and R_4 by a factor of ϵ^2. In this purely gravitational case the Ricci square R_2 remains in fact zero to all orders in ϵ.

[5] This relation in fact holds for any n to order ϵ^4.

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To leading order in ϵ, the equations (3.6)–(3.8) then reduces to the linear system

$$V_1 = \frac{H_1'}{n\cos^n u}\,, \qquad H_1'' + n^2 H_1 = 0\,. \tag{4.26}$$

Using the boundary conditions $V_1(0) = 0, H_1(0) = h_1$, one therefore obtains

$$H_1(u) = h_1 \cos nu\,, \qquad V_1(u) = -h_1 \frac{\sin nu}{\cos^n u}\,. \tag{4.27}$$

One may check that to this order, the curvature invariants $R_2 = (R_{\mu\nu})^2$ and $R_4 = (R_{\mu\nu\rho\sigma})^2$ remain zero. The length of the Killing vectors $\partial_{x,y}$ however is shifted to

$$\|\partial_{x,y}\|^2 = e^{-(U\mp V)/n} = \cos^2 u \pm \frac{\sin nu}{n\cos^{n-2} u}\epsilon h_1 + \mathcal{O}(\epsilon^2) \equiv 0\,. \tag{4.28}$$

For $n = 1$, the first order correction vanishes at $u = \pi/2$, hence the horizon remains to this order at $u = \pi/2$. For higher n however, the correction term blows up at $u = \pi/2$. Were the second order ϵ^2 term not more singular than the ϵ order term, the horizon would be shifted to

$$h_1 \epsilon v \sim \left(u - \frac{\pi}{2}\right)^{n-1} \qquad n \text{ even} \tag{4.29}$$

$$h_1 \epsilon v \sim n\left(u - \frac{\pi}{2}\right)^{n} \sim \qquad n \text{ odd}\,. \tag{4.30}$$

Unlike the Khan-Penrose case, it turns out that higher order term do change this behaviour, and lead to higher exponents.

At order ϵ^2, the equations of motion become linear with a source,

$$V_2 = \frac{H_2'}{n\cos^n u} + \frac{3\cos[(2n-1)u] + \cos[(2n+1)u]}{8\cos^{2n+1} u} h_1^2 \tag{4.31}$$

$$H_2'' + n^2 H_2 = -n\frac{(n-1)\sin[2nu] + 2n\sin[2(n-1)u]}{4\cos^{n+2} u} h_1^2 \tag{4.32}$$

$$M_2' = -\frac{n\sin[(2n-1)u] - (3n-2)\sin u}{4n\cos^{2n+1} u} h_1^2\,. \tag{4.33}$$

These equations are readily integrated for given values of n, using the boundary conditions $H_2(0) = h_2, V_2(0) = M_2(0) = 0$. The same pattern continues to hold to higher order, allowing us to get the solution to any order desired. Explicit solutions up to fourth order can be found in the appendix. We now briefly summarize our results.

In four dimensions, the collision of two collinear electromagnetic shock wave appears to be a rather smooth process. Indeed, when the two waves profiles are identical, the metric in the interaction region is given by the Bell-Szekeres solution, which is diffeomorphic to the Bertotti-Robinson or $AdS_2 \times S_2$ metric. It does however present a Killing horizon for the two Killing vectors $\partial_{x,y}$, at $u + v = \pi/2$. For more general profiles, we find that the curvature invariants remain finite,

$$R_2 = R_4 = 4h_1^2 + 8h_1 h_2 \epsilon v + (h_1^4 + 4h_2^2 + 4h_1 h_3)(\epsilon v)^2 + \mathcal{O}(\epsilon^3)\,. \tag{4.34}$$

To order ϵ^4, the norm of the Killing vectors is given by

$$\|\partial_x\|^2 = u^2 \pm h_1 u\epsilon v + \frac{h_1^2}{4}(\epsilon v)^2 + \frac{1}{3}h_1 h_2(\epsilon v)^3 \pm \frac{11h_1^2 h_2}{192u}(\epsilon v)^4 + \mathcal{O}(\epsilon^5)\,. \tag{4.35}$$

To leading order, the horizon therefore lies at

$$u - \frac{\pi}{2} = \pm\frac{1}{2}h_1\epsilon v\,. \tag{4.36}$$

As is well known however, the collision of an electromagnetic plane wave with a gravitational wave does lead to a curvature singularity. Indeed, if we allow for a non-zero $V^{(1)} = v_1 v + \frac{1}{2} v_2 v^2 + \cdots$ perturbation, the curvature invariants become

$$R_2 = R_4 = 4(h_1+v_1\tan u)^2 - \frac{4(h_1\cos u + v_1\sin u)((2h_2 - h_1 v_1)\cos u + 2v_2\sin u)}{\cos^2 u}\epsilon v + \mathcal{O}(\epsilon^2) \tag{4.37}$$

where we calculated but do not show the ϵ^2 term. As $u \to \pi/2$, the curvature blows up. The precise location of the singularity can be found by expanding $1/R_2$ and keeping the dominant term at each order,

$$R_2^{-1} = \frac{\Delta^2}{4v_1^2} - \frac{v_2\Delta^2}{2v_1^3}\epsilon v - \frac{3}{32\Delta^2}(\epsilon v)^2 + \mathcal{O}(\epsilon^3) \tag{4.38}$$

where we defined $\Delta = u - \pi/2$. The curvature singularity therefore appears at $v = \mathcal{O}(\Delta^2)$. For $h_1 = h_2 = 0$ this agrees with the explicit solution found by Griffiths [23].

In contrast, in 6 and higher dimensions, we observe that singularities are generically created, even for purely electromagnetic plane waves ($V^{(1)} = 0$). To order ϵ^3, the curvature invariants have a pole of high order at $u = \pi/2$. The order ϵ^4 contribution however is still more singular at $u = \pi/2$, implying that the singular locus is shifted away from $u = \pi/2$. Similarly, the norm of the Killing vectors diverges at higher order in ϵ, leading to a deviation from $u = \pi/2$. Using the same technique as in (4.22), we may derive the upper estimates for the locus of the Ricci square singularity, Riemann square singularity and of the Killing horizon:

$$R_2 = \infty: \quad v = \mathcal{O}\left[\left(u - \frac{\pi}{2}\right)^{\alpha}\right] \tag{4.39}$$

$$R_4 = \infty: \quad v = \mathcal{O}\left[\left(u - \frac{\pi}{2}\right)^{\beta}\right] \tag{4.40}$$

$$\|\partial_{x,y}\| = 0: \quad v = \mathcal{O}\left[\left(u - \frac{\pi}{2}\right)^{\gamma}\right]. \tag{4.41}$$

Our results may be summarized in the following table of upper critical exponents for $n = 1$ through $n = 4$,

n	α	β	γ
1	$n.a.$	$n.a.$	1
2	2	3	2
3	4	3	3
4	4	5	4

(4.42)

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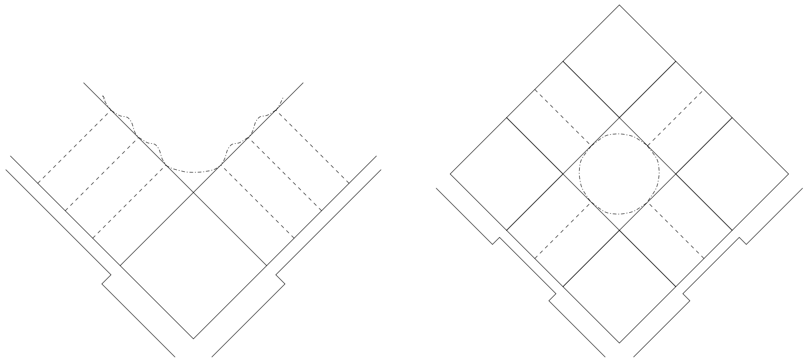

Figure 2: Left: collision of two electromagnetic shock waves for for $n > 1$. Right: collision of two trains of opposite shock waves, for special choice of pulse duration.

Generically, it therefore appears that, in contrast to the four-dimensional case, for $n > 1$ a line of curvature singularity is created tangentially to the light-cone at $v = 0$ at the fold singularities. One half of this line corresponds to a space-like singularity, while the other half is time-like. In order to determine the exact critical exponent of this singular line (instead of the upper bounds that we have derived), one would have to analyze the degree of divergence of the curvature invariants at $u = \pi/2$ to all orders in ϵ. In particular, in the absence of such an analysis, an essential singularity at $u = \pi/2$ cannot be ruled out. It would be interesting to check whether these conclusions remain valid in the case of non-collinear ($W \neq 0$) or non-purely electromagnetic ($V \neq 0$) waves.

Finally, we may obtain a perturbative solution describing the collision of two BFHP shock waves in type-IIB supergravity by specializing our analysis to $n = 4$ and $H^{(1)}(v) = H^{(0)}(v)$, i.e.

$$h_1 = 2n\,, \qquad h_2 = 0\,, \qquad h_3 = -2n^2\,, \qquad h_4 = 0\,, \ldots \tag{4.43}$$

The perturbative analysis performed above then applies in the vicinity of either of the light-cone axes, with the added simplification that all even coefficients $h_2, h_4, h_6 \ldots$ vanish. It is therefore tempting to conjecture that the spacelike singularity emanating from $u = \pi/2, v = 0$ joins smoothly onto the one emanating from $u = 0, v = \pi/2$, as depicted in figure 2. Furthermore, it is plausible that the time-like singularity emanating from $u = (2p-1)\pi/2$ merges on to the space-like singularity emanating from $u = (2p+1)\pi/2$. Assuming this is correct, an interesting configuration arises when the incoming waves involves a succession of two shock waves of opposite amplitude separated by a critical time π: using time reversal invariance, it is easy to see that the singular locus will form a closed line inside the interaction region.

5. Discussion

In this note we have studied colliding plane wave solutions of $2n+2$-dimensional Einstein gravity with a $n+1$-form field strength. A natural ansatz (3.2)–(3.3) was shown to lead to the same equations as in ordinary four-dimensional Einstein-Maxwell gravity. By upgrading known four-dimensional solutions, we were able to construct exact solutions in $2n+2$

dimensions. In general however, the boundary conditions suitable in dimension 4 are not appropriate in higher dimensions. In the second part of the paper, we therefore developed a perturbative scheme in order to determine CPW solutions with specified boundary conditions, in the vicinity of the light-cone.

The general conclusion of this study is that, just as in four dimensions, space-like singularities usually develop in the interaction region, emanating from the fold singularities in the incoming plane waves. An exception is the case of four-dimensional purely electromagnetic plane waves, which only develop a Killing horizon in region F. Switching on an arbitrarily small gravitational perturbation however immediately leads to a space-like singularity, as was already observed in the context of Kerr black holes [4]. In higher dimensions, purely electromagnetic plane wave collisions already display space-like singularities, as well as Killing horizons. By specializing to identical incoming waves, we were able to obtain the metric of the higher dimensional version of the Khan-Penrose and Bell-Szekeres CPW solutions in the vicinity of the light-cone. For $n = 4$, the latter describes the collision of two BFHP plane waves in type-IIB string theory, or rather of their shock wave generalization.

As in earlier studies of plane wave collisions, it is worth stressing that the creation of singularities for arbitrary small wave amplitude is tied to the plane wave symmetry, and would presumably acquire a threshold in the case of more realistic pp-waves.

Another general result is that the space-like singularity generally appears in combination with a time-like singularity, the two of them meeting at the fold singularity. In earlier studies of the four-dimensional problem, it has generally be assumed that the region behind the fold singularity should be excised, thereby getting rid of the time-like singularity in the forward region. However, in view of recent developments in the understanding of the asymptotic structure of plane waves and holography [28]–[31], this prescription seems hard to justify. Surely, in the absence of the counter-propagating wave, the fold singularity is simply an artefact of the Rosen coordinate system. Even in the presence of the second plane wave, there exist null geodesics which can cross the fold singularity and possibly reach the time-like singularity without encountering any horizon. This does not necessary pose a conflict with cosmic censorship, as the latter may not apply in non-asymptotically flat geometries.

In contrast to pp-wave backgrounds, colliding plane wave geometries in string theory receive α' and g_s corrections. It would therefore be very interesting to find exact conformal field theory descriptions of such geometries. The Bertotti-Robinson $AdS_2 \times S_2$ geometry or the Wess-Zumino-Witten model $\mathrm{Sl}(2) \times \mathrm{SU}(2)/\mathbb{R} \times \mathbb{R}$ are exact solutions of string theory [8], but it is not known how to impose the junction conditions at $uv = 0$ while preserving conformal invariance. On the other hand, string probes propagating in a pp-wave background are just another example of plane wave collisions, and it is a very important problem to understand their backreaction on the plane wave background.

In this respect, let us note that there exists a different perturbative scheme than the one considered in section 4, where one expands in the amplitude of the perturbation $g^{(1)}(v)$ rather than its gradient, i.e. define $\mathrm{U}(u,v) = -\log\left[f^{(0)} - 1/2 + \epsilon^2(g^{(1)}(\epsilon v) - 1/2)\right]$. At each order in ϵ the profiles are general functions of (u,v) rather than of u only. Assuming that the perturbation has compact support along v, one may compute perturbatively its effect on the background right-moving wave. To leading order in ϵ, the equations (3.6)–(3.8) are total derivatives with respect to v; after integration, they become ordinary differential equations with respect to u only, with initial conditions set by the left-moving profile $H^{(1)}, V^{(1)}$ at every value of v. This reduction to a one-dimensional dynamical system is very reminiscent to the dimensional reduction that takes place near a space-like singularity, however the instability of null singularities render this observation less useful.

To conclude, plane wave collisions are an inexhaustible source of space-like singularities. It would be very useful to develop a holographic description of them, as it may provide some insight into the dynamics of strings near a cosmological singularity.

Acknowledgments

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A. Perturbative expansions — the impulsive gravitational case

A.1 $n = 1$

At second order,

$$H_2(u) = h_2\sqrt{\frac{1+u}{1-u}} + h_1 v_1 \frac{1+u-\sqrt{1-u^2}}{2(1-u)} \tag{A.1}$$

$$V_2(u) = \frac{u v_1^2}{2(1-u^2)} + \frac{v_2}{\sqrt{1-u^2}} \tag{A.2}$$

$$M_2(u) = \frac{u^2(v_1^2 - h_1^2)}{4(1-u^2)} + \frac{u v_2}{\sqrt{1-u^2}}\,. \tag{A.3}$$

At third order, for vanishing h_i,

$$V_3(u) = -\frac{2u\sqrt{1-u^2}v_1 v_2 + u^2\left({v_1}^3 - 2v_3\right) + 2v_3}{2\left(1-u^2\right)^{\frac{3}{2}}} \tag{A.4}$$

$$M_3(u) = \frac{u\left(\left(1+u^2\right){v_1}^3 + 3u\sqrt{1-u^2}v_1 v_2 + 4\left(1-u^2\right)v_3\right)}{4(1-u^2)^{\frac{3}{2}}}\,. \tag{A.5}$$

The fourth order correction was computed but is not displayed here. It allows us to compute the curvature invariant R_4 through order ϵ^2,

$$
\begin{aligned}
R_4 \;=\; & \frac{6{v_1}^2}{(1-u^2)^3} + 9v_1(\epsilon v)\frac{3u\sqrt{1-u^2}{v_1}^2 + 2v_2 - 2u^2 v_2}{(1-u^2)^4} + \\
& +3(\epsilon v)^2\frac{(-1+43u^2)v_1^4 + 55u\sqrt{1-u^2}v_1^2 v_2 + 6(1-u^2)v_2^2 + 8(1-u^2)v_1 v_3}{2(1-u^2)^4} + \mathcal{O}(\epsilon^3)\,.
\end{aligned}
\tag{A.6}
$$

Its inverse can be expanded around $u = \pi/2$, keeping the most dominant term at each order in ϵ,

$$
R_4^{-1} = -\frac{3v_1^2}{4\Delta^3} + \frac{27v_1^3}{8\sqrt{2}\Delta^{7/2}}\epsilon v + \frac{63v_1^4}{16\Delta^4}(\epsilon v)^2 + \mathcal{O}(\epsilon^3)\,. \tag{A.7}
$$

The space-like singularity therefore lies at

$$
\epsilon v = \mathcal{O}(\Delta^{1/2})\,. \tag{A.8}
$$

Similarly, the length of the Killing vectors reads, keeping the most dominant term at each order in ϵ,

$$
\|\partial_x\|^2 = \Delta^2 + \frac{v_1\Delta^{3/2}}{\sqrt{2}}\epsilon v - \frac{v_2\Delta^{3/2}}{2\sqrt{2}}(\epsilon v)^2 + \frac{v_1^3\Delta^{1/2}}{8\sqrt{2}}(\epsilon v)^3 + \frac{v_1^4}{64}(\epsilon v)^4\,. \tag{A.9}
$$

The Killing horizon thus lies at

$$
\epsilon v = \mathcal{O}(\Delta^{1/2})\,. \tag{A.10}
$$

A.2 $n = 2$

For conciseness, for $n \geq 2$ we restrict to the purely gravitational case, $h_i = 0$. At second order in ϵ,

$$
V_2(u) \;=\; \frac{2(4v_2 + u({v_1}^2 - 4uv_2)) - (1-u^2){v_1}^2(\log\frac{1-u}{1+u})}{8(-1+u^2)^2} \tag{A.11}
$$

$$
M_2(u) \;=\; \frac{4u(u{v_1}^2 - 8v_2) + 4(4v_2 + u({v_1}^2 - 4uv_2))\log(\frac{1-u}{1+u}) + (-1+u^2){v_1}^2\log(\frac{1-u}{1+u})^2}{64(-1+u^2)}\,. \tag{A.12}
$$

We computed but do not display the third and fourth order contributions. This allows to compute the curvature invariant R_4 through order ϵ^2,

$$
\begin{aligned}
R_4 \;=\; & \frac{2(3+u^2)v_1^2}{(1-u^2)^4} - (\epsilon v)\frac{v_1}{4(1-u^2)^5} + \\
& + \Big[-2u\left(31+9u^2\right)v_1^2 24\left(-3+2u^2+u^4\right)v_2 + \\
& \quad + 9\left(-3+2u^2+u^4\right)v_1^2\log\left(\frac{1+u}{1-u}\right)\Big] + \mathcal{O}(\epsilon^2)\,.
\end{aligned}
\tag{A.13}
$$

Expanding the inverse around $\pi/2$

$$
R_4^{-1} = \frac{2\Delta^4}{v_1^2} + \frac{5\Delta^3}{2v_1}(\epsilon v) + \frac{87}{64}\Delta^2(\epsilon v)^2 + \mathcal{O}(\epsilon^3) \tag{A.14}
$$

we find that the curvature singularity lies at

$$\epsilon v = \mathcal{O}(\Delta)\,. \tag{A.15}$$

The length of the Killing vectors reads

$$\|\partial_x\|^2 = \Delta^2 + \frac{v_1\Delta}{4}\epsilon v - \frac{8v_2 - v_1^2}{64}\Delta(\epsilon v)^2 - \frac{v_1^3}{384\Delta}(\epsilon v)^3 - \frac{v_1^4}{6144\Delta^2} \tag{A.16}$$

hence the Killing horizon lies at

$$\epsilon v = \mathcal{O}(\Delta)\,. \tag{A.17}$$

A.3 $n = 3$

At second order,

$$V_2(u) = \frac{u\left(-3+2u^2\right){v_1}^2}{6(-1+u^2)^3} + \frac{v_2}{(1-u^2)^{\frac{3}{2}}} \tag{A.18}$$

$$M_2(u) = \frac{u\left(uv_1^2\left(3-6u^2+2u^4\right)+12v_2\left(-3+2u^2\right)(1-u^2)^{3/2}\right)}{36\,(1-u^2)^3}\,. \tag{A.19}$$

We computed but do not display the third and fourth order contributions. This allows to compute the curvature invariant R_4 through order ϵ^2,

$$\begin{aligned} R_4 = {} & \frac{2(2u^2+3)v_1^2}{(1-u^2)^5} - (\epsilon v)\frac{v_1}{3(1-u^2)^7} \times \\ & \times \left[u\sqrt{1-u^2}\left(89-36u^4\right)v_1^2 - 18\left(1-u^2\right)^2\left(3+2u^2\right)v_2\right] + \mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{A.20}$$

Expanding its inverse,

$$R_4^{-1} = -\frac{16\Delta^5}{5v_1^2} - \frac{10\sqrt{2}\Delta^{7/2}}{3v_1}(\epsilon v) + \frac{309\Delta^2}{50}(\epsilon v)^2 + \mathcal{O}(\epsilon^3) \tag{A.21}$$

we find a curvature singularity at

$$\epsilon v = \mathcal{O}(\Delta^{3/2})\,. \tag{A.22}$$

The length of the Killing vectors reads

$$\|\partial_x\|^2 = \Delta^2 - \frac{v_1\Delta^{1/2}}{6\sqrt{2}}\epsilon v - \frac{v_1^2}{96}(\epsilon v)^2 - \frac{v_1^3}{5184\sqrt{2}\Delta^{5/2}}(\epsilon v)^3 + \frac{v_1^4}{248832\Delta^4}(\epsilon v)^4 \tag{A.23}$$

hence the Killing horizon lies at

$$\epsilon v = \mathcal{O}(\Delta^{3/2})\,. \tag{A.24}$$

A.4 $n = 4$

At second order,

$$V_2(u) = \frac{-2u\left(-5+3u^2\right){v_1}^2 + 32\left(1-u^2\right)^2 v_2 + 3\left(1-u^2\right)^2 {v_1}^2 \log(\frac{1+u}{1-u})}{32(1-u^2)^4} \tag{A.25}$$

$$M_2(u) = \frac{4u\left(u\left(-23+42u^2-39u^4+12u^6\right){v_1}^2 - 32\left(1-u^2\right)^2\left(-5+3u^2\right)v_2\right)}{1024(1-u^2)^4} + \tag{A.26}$$
$$+\frac{3\left(4u\left(-5+3u^2\right)v_1^2 - 64\left(1-u^2\right)^2 v_2 + 3\left(1-u^2\right)^2 v_1^2\left(\log\frac{1-u}{1+u}\right)\right)\log\frac{1-u}{1+u}}{1024(1-u^2)^2}.$$

We computed but do not display the third and fourth order contributions. This allows to compute the curvature invariant R_4 through order ϵ^2,

$$\begin{aligned} R_4 = {} & \frac{6v_1^2(1+u^2)}{(1-u^2)^6} + \frac{3v_1(\epsilon v)}{16(1-u^2)^8} \times \\ & \times\left[-2u\left(-53-18u^2+27u^4\right){v_1}^2 + \left(1-u^2\right)^2\left(1+u^2\right)\times \right. \\ & \left. \times\left(96v_2 + 27{v_1}^2\log\left(\frac{1+u}{1-u}\right)\right)\right] + \mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{A.27}$$

Expanding its inverse,

$$R_4^{-1} = \frac{16\Delta^6}{3v_1^2} - \frac{11v_1\Delta^4}{6v_1}\epsilon v + \frac{41v_1^2\Delta^2}{192}(\epsilon v)^2 + \mathcal{O}(\epsilon^3) \tag{A.28}$$

we find that the singularity lies at

$$\epsilon v = \mathcal{O}(\Delta^2)\,. \tag{A.29}$$

The length of the Killing vectors reads

$$\|\partial_x\|^2 = \Delta^2 - \frac{v_1}{16}\epsilon v - \frac{v_1^2}{512\Delta}(\epsilon v)^2 + \frac{v_1^3}{8192\Delta^3}(\epsilon v)^3 + \frac{9v_1^4}{2621440\Delta^5}(\epsilon v)^4 \tag{A.30}$$

hence the Killing horizon lies at

$$\epsilon v = \mathcal{O}(\Delta^2)\,. \tag{A.31}$$

B. Perturbative expansions — the electromagnetic case

B.1 $n = 1$

At second order,

$$H_2(u) = h_2\cos u - \frac{1}{2}h_1^2\sin u\,, \qquad V_2(u) = -h_2\tan u\,, \qquad M_2(u) = 0\,. \tag{B.1}$$

At third order,

$$
\begin{aligned}
H_3(u) &= h_3 \cos u - \frac{3}{2} h_1 h_2 \sin u && \text{(B.2)}\\
V_3(u) &= -\frac{\tan u}{8\cos^2 u}\left[5h_1^3 + 4h_3 + (h_1^3 + 4h_3)\cos 2u\right] && \text{(B.3)}\\
M_3(u) &= \frac{1}{4} h_1 h_2 \tan^2 u && \text{(B.4)}
\end{aligned}
$$

At fourth order,

$$
\begin{aligned}
H_4(u) &= \frac{1}{8}\left[\left(-5h_1^2 h_2 + 8h_4\right)\cos u - \left(3h_1^4 + 12h_2^2 + 16h_1h_3\right)\sin u + 5h_1^2 h_2 \sec u\right] && \text{(B.5)}\\
V_4(u) &= -\frac{\tan u}{16\cos^2 u}\left(45h_1^2 h_2 + 8h_4 + \left(7h_1^2 h_2 + 8h_4\right)\cos 2u\right) && \text{(B.6)}\\
M_4(u) &= \frac{1}{8}\left(h_1^4 + 2h_2^2 + 4h_1h_3\right)\tan^2 u\,. && \text{(B.7)}
\end{aligned}
$$

The Ricci square reads

$$
R_2 = 4h_1^2 + 8h_1h_2\epsilon v + (h_1^4 + 4h_2^2 + 4h_1h_3)(\epsilon v)^2 + \mathcal{O}(\epsilon^3)\,. \qquad \text{(B.8)}
$$

One may check that this is also equal to the Riemann tensor square $R_4 = R_2$.

B.2 $n = 2$

At second order,

$$
\begin{aligned}
H_2(u) &= h_2 \cos 2u - h_1^2 \tan u \cos 2u && \text{(B.9)}\\
V_2(u) &= -2h_2 \tan u && \text{(B.10)}\\
M_2(u) &= h_1^2 \frac{\tan^2 u}{32\cos^2 u}(5\cos 2u - 1)\,. && \text{(B.11)}
\end{aligned}
$$

At third order,

$$
\begin{aligned}
H_3(u) &= h_3 \cos 2u - h_1 \frac{\left(9h_2\left(\cos u + \cos 3u\right) + 2h_1^2\left(3\sin u - 2\sin 3u\right)\tan(u)\right)}{6\cos u} && \text{(B.12)}\\
V_3(u) &= \frac{1}{6}\left(4\left(h_1^3 - 3h_3\right) - h_1^3 \mathrm{sec}(u)^2\left(6 + \sec^2 u\right)\right)\tan u && \text{(B.13)}\\
M_3(u) &= \frac{3h_1h_2\left(1 + 3\cos(2u)\right)\tan^2 u}{16\cos^2 u}\,. && \text{(B.14)}
\end{aligned}
$$

At fourth order,

$$
\begin{aligned}
H_4(u) &= h_4 \cos 2u + \frac{\tan(u)}{12} \times\\
&\quad \times\Big[4\left(7h_1^4 - 9h_2^2 - 12h_1h_3\right)\cos(2u) + h_1^4\left(-72 + 40\mathrm{sec}(u)^2 + \mathrm{sec}(u)^4\right) +\\
&\quad + h_1^2 h_2\left(-14 + 101\cos(2u)\right)\tan(u)\Big] && \text{(B.15)}\\
V_4(u) &= \frac{\tan u}{12}\left(58h_1^2 h_2 - 24h_4 - h_1^2 h_2\left(43 + 36\cos(2u)\right)\sec^4 u\right) && \text{(B.16)}\\
M_4(u) &= -\frac{\sec^2(u) - 1}{1152}\Big(529h_1^4 - 1296h_2^2 - 1872h_1h_3 + \left(-815h_1^4 + 432h_2^2 + 432h_1h_3\right)\times\\
&\quad \times \mathrm{sec}(u)^2 + h_1^4\left(179 + 17\cos(2u)\right)\mathrm{sec}(u)^6\Big)\,. && \text{(B.17)}
\end{aligned}
$$

The Ricci square curvature invariant reads

$$
\begin{aligned}
R_2 &= 6\left[(h_1+h_2\epsilon v)^2+h_1h_3(\epsilon v)^2\right]\frac{\cos^2 2u}{\cos^4 u} \\
&\quad -h_1^4(\epsilon v)^2\frac{\cos 2u}{32\cos^6 u}\left(1-90\cos 2u-31\cos 4u\right)\tan^2 u+\mathcal{O}(\epsilon^3)
\end{aligned} \tag{B.18}
$$

In order to find the location of the singularity, we expand $1/R_2$ around $u=\pi/2$, keeping the dominant term at each order in ϵ

$$
R_2^{-1} = \frac{1}{6h_1^2}\Delta^4 - \frac{h_2}{3h_1^3}\Delta^4\ \epsilon v - \frac{5}{96}(v\epsilon)^2 + \mathcal{O}(\epsilon^3)\,. \tag{B.19}
$$

At this order, the Ricci curvature singularity therefore lies at $v=\mathcal{O}(\Delta^2)$. As explained in the text, this is only an upper estimate, as higher order ϵ-corrections to R_2 may be equally or more singular at $u=\pi/2$. We now turn to the Riemann square curvature invariant,

$$
\begin{aligned}
R_4 = &\frac{h_1^2\left(7+3\cos(4u)\right)\sec(u)^4}{2} + h_1h_2\left(7+3\cos(4u)\right)\sec(u)^4\epsilon v \\
&+\frac{1}{48}(v\epsilon)^2\Big[12\left(-31h_1^4+48h_2^2+48h_1h_3\right)+12\left(55h_1^4-48h_2^2-48h_1h_3\right)\sec(u)^2 - \\
&\quad -5\left(139h_1^4-48h_2^2-48h_1h_3\right)\sec(u)^4+604h_1^4\sec^6 u-263h_1^4\sec^8 u \\
&\quad +90h_1^4\sec^{10} u\Big]+\mathcal{O}(\epsilon^3)\,.
\end{aligned} \tag{B.20}
$$

At each order in ϵ, the most dominant terms are

$$
R_4^{-1} = \frac{1}{4h_1^2}\Delta^4 - \frac{2h_2}{5h_1^3}\Delta^4\ \epsilon v - \frac{3}{40\Delta^2}(\epsilon v)^2 + \mathcal{O}(\epsilon^3)\,. \tag{B.21}
$$

The Riemann square singularity is therefore at $v=\mathcal{O}(\Delta^3)$. The length of the Killing vector reads

$$
\|\partial_{x,y}\|^2 = \Delta^2 \pm h_1\Delta\epsilon v - \frac{h_1^2}{16\Delta^2}(\epsilon v)^2 \mp \frac{h_1^3}{144\Delta^3}(\epsilon v)^3 - \frac{h_1^4}{512\Delta^6}(\epsilon v)^4 + \mathcal{O}(\epsilon^5)\,. \tag{B.22}
$$

The Killing horizons are therefore at $v=\mathcal{O}(\Delta^2)$.

B.3 $n=3$

At second order,

$$
H_2(u) = h_2\cos 3u - \frac{1}{2}h_1^2\sec^2 u(\sin u - \sin 3u + \sin 5u) \tag{B.23}
$$

$$
V_2(u) = -h_2\frac{\sin 3u}{\cos^3 u} \tag{B.24}
$$

$$
M_2(u) = h_1^2\frac{\tan^2 u}{288\cos^4 u}(-3+56\cos 2u+43\cos 4u)\,. \tag{B.25}
$$

At third order,

$$\begin{aligned}
H_3(u) &= h_3\cos 3u - \frac{h_1\sin 3u}{240\cos^5 u}[360h_2\cos^3 u\,(2\cos 2u - 1) + \\
&\qquad + h_1^2\,(-80\sin u + 95\sin 3u - 89\sin 5u)] \\
V_3(u) &= \frac{1}{60}\tan u[464h_1^3 - 240h_3 + \sec^2 u\left(-836h_1^3 + 60h_3\right) + \\
&\qquad + h_1^3\sec^4 u\left(450 - 103\sec^2 u + 10\sec^4 u\right)] \\
M_3(u) &= \frac{h_1h_2\,(33 + 104\cos 2u + 67\cos 4u)\tan^2 u}{144\cos^4 u}\,.
\end{aligned} \tag{B.26}$$

At fourth order,

$$\begin{aligned}
H_4(u) &= h_4\cos 3u + 216h_1^2h_2\cos u + \\
&\quad + \frac{1}{120}[(-4312h_1^2h_2 + 120h_4)\cos 3u - 120h_4\cos 4u - \\
&\qquad - 48(536h_1^4 - 45h_2^2 - 60h_1h_3)\sin u + 16(176h_1^4 - 45h_2^2 - 60h_1h_3)\sin 3u - \\
&\qquad - 138h_1^4\sec^7 u\tan u + \sec^3 u(8359h_1^2h_2 - 10396h_1^4\tan u) + \\
&\qquad + \sec^5 u(-807h_1^2h_2 + 1927h_1^4\tan u) - \\
&\qquad - 12\sec u(2430h_1^2h_2 + (-2156h_1^4 + 45h_2^2 + 60h_1h_3)\tan u)]
\end{aligned} \tag{B.27}$$

$$\begin{aligned}
V_4(u) &= \frac{4}{15}(179h_1^2h_2 - 15h_4) + \frac{\tan u}{120\cos^2 u}\times \\
&\quad \times[-10072h_1^2h_2 + 120h_4 + h_1^2h_2(5400\sec^2 u - 1181\sec^4 u + 95\sec(u)^6)]
\end{aligned} \tag{B.28}$$

$$\begin{aligned}
M_4(u) &= \frac{1-\sec^2(u)}{16200}[4(31049h_1^4 - 15075h_2^2 - 20475h_1h_3)\times \\
&\qquad \times\sec^2 u(-255964h_1^4 + 36900h_2^2 + 47700h_1h_3) + \\
&\qquad + \sec(u)^4(178196h_1^4 + 450h_2^2 - 900h_1h_3) + \\
&\qquad + 9h_1^4\sec(u)^6(-6301 + 1184\sec(u)^2 + 50\sec(u)^4)]\,.
\end{aligned} \tag{B.29}$$

The Ricci square reads

$$\begin{aligned}
R_2 &= 8[(h_1 + h_2\epsilon v)^2]\frac{(1-2\cos 2u)^2}{\cos^4 u} - \frac{2}{45}h_1(\epsilon v)^2(3\sec^2 u - 4)\times \\
&\quad \times[-2252h_1^3 + 720h_3 + \sec^2 u(3849h_1^3 - 540h_3) + \\
&\qquad + 2h_1^3\sec^4 u(-585 - 362\sec^2 u + 141\sec^4 u)] + \mathcal{O}(\epsilon^3)\,.
\end{aligned} \tag{B.30}$$

Keeping the most singular terms at $\pi/2$, we find

$$R_2^{-1} = \frac{1}{72h_1^2}\Delta^4 + \frac{47}{6480\Delta^2}(\epsilon v)^2 + \mathcal{O}(\epsilon^3)\,. \tag{B.31}$$

The Ricci singularity is therefore at $v = \mathcal{O}(\Delta^3)$. The Riemann square reads

$$\begin{aligned}
R_4 \;=\; & \frac{h_1^2}{3\cos^8 u}(8 + 2\cos 2u + 7\cos 4u + 2\cos 6u + \cos 8u) + \\
& + \frac{2h_1h_2}{3\cos^8 u}\left(-\cos 2u + 8 + 10\cos(4u) + 2\cos 6u + \cos 8u\right)\epsilon v + \\
& + \frac{1}{135}(\epsilon v)^2 [32(-563h_1^4 + 180h_2^2 + 180h_1h_3) + 48(923h_1^4 - 180h_2^2 - 180h_1h_3)\sec^2 u - \\
& \qquad - 6(11261h_1^4 - 1080h_2^2 - 1260h_1h_3)\sec^4 u + \\
& \qquad + (92029h_1^4 - 3510h_2^2 - 4860h_1h_3)\sec^6 u \\
& \qquad - 18(4267h_1^4 - 45h_2^2 - 60h_1h_3)\sec^8 u + 30690h_1^4 \sec^{10} u - \\
& \qquad - 5040h_1^4 \sec^{12} u + 420h_1^4 \sec^{14} u] + \mathcal{O}(\epsilon^3)\,.
\end{aligned} \tag{B.32}$$

Expanding,

$$R_4^{-1} = \frac{1}{4h_1^2}\Delta^8 - \frac{3}{h_2}4h_1^3\Delta^8\ \epsilon v - \frac{7}{36}\Delta^2(\epsilon v)^2 + \mathcal{O}(\epsilon^3)\,. \tag{B.33}$$

It is singular at $v \sim \mathcal{O}(\Delta^3)$. The length of the Killing vector reads

$$\|\partial_{x,y}\|^2 = \Delta^2 \pm \frac{h_1}{3\Delta}\epsilon v + \frac{h_1^2}{36\Delta^4}(\epsilon v)^2 \pm \frac{h_1^3}{162\Delta^7}(\epsilon v)^3 + \frac{5h_1^4}{3888\Delta^4}(\epsilon v)^4 + \mathcal{O}(\epsilon^5)\,. \tag{B.34}$$

The Killing horizons are therefore at $v \sim \mathcal{O}(\Delta^3)$.

B.4 $n = 4$

At second order,

$$H_2(u) = h_2\cos 4u + 2\frac{(\sin u - \sin 3u)\cos 4u}{\cos^3 u}h_1^2 \tag{B.35}$$

$$V_2(u) = 2\frac{(\sin u - \sin 3u)}{\cos^3 u}h_2 \tag{B.36}$$

$$M_2(u) = \frac{\tan^2 u(22 + 439\cos 2u + 442\cos 4u + 249\cos 6u)}{2048\cos^6 u}h_1^2\,. \tag{B.37}$$

At third order,

$$\begin{aligned}
H_3(u) \;=\; & \frac{\cos 4u}{32\cos^6 u}(48h_1^3 + 10h_3 - 3(24h_1^3 - 5h_3)\cos 2u + 6(8h_1^3 + h_3)\cos 4u - 24h_1^3\cos 6u + \\
& + h_3\cos 6u - 12h_1h_2\sin 2u - 24h_1h_2\sin 4u - 12h_1h_2\sin 6u)
\end{aligned} \tag{B.38}$$

$$V_3(u) = 4\left(-2 + \sec^2 u\right)\left(-8h_1^3 + h_3 + h_1^3\left(3 + 3\cos 2u + 2\cos 4u\right)\sec^6 u\right)\tan u \tag{B.39}$$

$$M_3(u) = \frac{h_1h_2}{1024\cos^6 u}\left(278 + 823\cos 2u + 698\cos 4u + 377\cos 6u\right)\tan^2 u\,. \tag{B.40}$$

At fourth order,

$$
\begin{aligned}
H_4(u) \;&=\; h_4 \cos 4u + \frac{\cos 4u}{\cos^3 u}(3h_2^2(\sin u - \sin 3u)) + \\
&\quad + 3h_1^4 \frac{\cos 4u}{\cos^9 u}(\sin u - \sin 3u)^3 + 2h_1 h_3 \sec^3 u(\sin u - \sin 3u + \sin 5u - \sin 7u) + \\
&\quad + \frac{h_1^2 h_2 \tan^2 u}{56 \cos^4 u}(533 + 72\cos 2u + 1038\cos 4u + 8\cos 6u + 505\cos 8u) \qquad \text{(B.41)} \\
V_4(u) \;&=\; 4h_4 \tan u(\tan^2 u - 1) + \qquad \text{(B.42)} \\
&\quad + \frac{h_1^2 h_2 \tan u}{224 \cos^{10} u}(8 - 166\cos 2u + 64\cos 4u + 197\cos 6u + 8\cos 8u + 169\cos 10u) \\
M_4(u) \;&=\; h_1 h_3(\sec^2 u - 1)\frac{505 - 519\sec^2 u + 121\sec^4 u - 7\sec^6 u}{32} + \\
&\quad + h_1^4\left(\frac{196643}{2048} - 384\sec^2 u + 624\sec^4 u - 528\sec^6 u + \right. \\
&\qquad \left. + \frac{503857}{2048}\sec^8 u - 60\sec^{10} u + 6\sec^{12} u - \frac{21\sec^{16} u}{512}\right) + \\
&\quad + \frac{h_2^2\,(278 + 823\cos 2u + 698\cos 4u + 377\cos 6u)\tan^2 u}{1024\cos^6 u}\,. \qquad \text{(B.43)}
\end{aligned}
$$

The Ricci square reads

$$
\begin{aligned}
R_2 \;&=\; 10\left[(h_1 + h_2\epsilon v)^2 + h_1 h_3(\epsilon v)^2\right]\frac{\cos^2 4u}{\cos^8 u} - 5h_1^4(\epsilon v)^2\frac{\cos^2 4u}{4096\cos^{16} u} \times \\
&\quad \times(-1141 + 1656\cos 2u - 2244\cos 4u + 1992\cos 6u + 761\cos 8u) + \mathcal{O}(\epsilon^3)\,. \qquad \text{(B.44)}
\end{aligned}
$$

Expanding

$$
R_2^{-1} = \frac{1}{10h_1^2}\Delta^8 - \frac{h_2}{5h_1^3}\Delta^8\ \epsilon v - \frac{49}{640}(\epsilon v)^2 + \mathcal{O}(\epsilon^3)\,. \qquad \text{(B.45)}
$$

It diverges at $v \sim \mathcal{O}(\Delta^4)$. The Riemann square reads

$$
\begin{aligned}
R_4 \;=\;& \frac{77 - 96\cos(2u) + 48\cos(4u) + 5\cos(8u)}{4\cos^8 u}h_1^2 + \\
&+ \frac{101 - 144\cos(2u) + 72\cos(4u) + 5\cos(8u)}{2\cos^8 u}h_1 h_2 \epsilon v + \\
&+ \frac{1}{128}(\epsilon v)^2\Big[320(-761h_1^4 + 64h_2^2 + 64h_1 h_3) + 640(1273h_1^4 - 64h_2^2 - 64h_1 h_3)\sec^2 u - \\
&\qquad - 16(104105h_1^4 - 2752h_2^2 - 3136h_1 h_3)\sec^4 u + \\
&\qquad + 16(177121h_1^4 - 2048h_2^2 - 2624h_1 h_3)\sec^6 u - \qquad \text{(B.46)} \\
&\qquad - (3232777h_1^4 - 10304h_2^2 - 13376h_1 h_3)\sec^8 u + 2103680h_1^4\sec^{10} u - \\
&\qquad - 714416h_1^4\sec^{12} u + 108848h_1^4\sec^{14} u - 5083h_1^4\sec^{16} u + 252h_1^4\sec^{18} u\Big]\,.
\end{aligned}
$$

Expanding

$$
R_4^{-1} = \frac{2}{113h_1^2}\Delta^8 - \frac{644h_2}{113^2 h_1^3}\epsilon v - \frac{63}{8\cdot 113^2\Delta^2}(\epsilon v)^2 + \mathcal{O}(\epsilon^3) \qquad \text{(B.47)}
$$

we see that the R_4 curvature invariant is singular at $v \sim \mathcal{O}(\Delta^5)$. The length of the Killing vector reads

$$\|\partial_{x,y}\|^2 = \Delta^2 \pm \frac{h_1}{\Delta}\epsilon v - \frac{h_1^2}{64\Delta^6}(\epsilon v)^2 \mp \frac{h_1^3}{64\Delta^9}(\epsilon v)^3 - \frac{3h_1^4}{8192\Delta^4}(\epsilon v)^4 + \mathcal{O}(\epsilon^5)\,. \tag{B.48}$$

The Killing horizons are therefore at $v \sim \mathcal{O}(\Delta^4)$.

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